11. U and H, the body's single momentum and momentum couple, may be found without difficulty in the ordinary way. In the case of a rigid body moving about a fixed point O, if A, B, C be the moments of inertia, and ω_x, ω_y, ω_z be the angular velocities of rotation about the principal axes O_x, O , O_z, then H is the resultant of $A\omega_x$, $B\omega_y$, $C\omega_z$. Let then Λ denote the instantaneous axis and angular velocity of rotation, then D_t (H) is, by one of our fundamental theorems, equivalent to det (Λ, H), together with

$$\frac{d}{dt}(A\omega_x) \parallel \text{to } O_x, \quad \frac{d}{dt}(B\omega_y) \parallel \text{to } O_y, \quad \frac{d}{dt}(C\omega_z) \parallel \text{to } O_z,$$

and by D'Alembert's principle D_t (H) is the axis of the resultant couple formed by transferring to O all the external forces.

12. The last theorem includes Euler's equations, and the different extensions which those have of late received. I have attempted to show that, in solving mechanical problems, the above theorem will be generally found more useful than any of those equations. Moreover, it serves to explain the geometrical reason, independently of Euler's equations, why the results are so much simplified in the case of two of the principal moments of inertia being equal to one another.

13. In order to illustrate how advantageously and how completely the most complicated formulæ of dynamics may be interpreted, I have given a direct analytical proof of Euler's equations, and have then shown that the consideration of each step of the analytical work leads to an extremely short demonstration of the same equations by means of the theory of the differential coefficients and determinants of lines.

XV. "On the Theory of Probabilities." By GEORGE BOOLE, Esq., F.R.S. Received May 21, 1862.

(Abstract.)

This paper has for its object the investigation of the general analytical conditions of a method for the solution of questions in the Theory of Probabilities, which was published in a work entitled "An Investigation of the Laws of Thought, &c."*

The application of the method to particular problems has been

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* London, Walton and Maberly, 1854.

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illustrated in the work referred to, and more fully in a Memoir published in the Transactions of the Royal Society of Edinburgh, entitled "On the Application of the Theory of Probabilities to the Question of the Combination of Testimonies or Judgments." The latter contains also the foundations of the general analytical theory of the method. But the complete development of that theory is attended with mathematical difficulties which I have only just succeeded in overcoming.

A correspondence on the subject between Mr. Cayley and myself also appears in the current (May) Number of the 'Philosophical Magazine,' and I owe it to Mr. Cayley that these further researches, of the results of which an account will here be given, were undertaken.

I shall make but few remarks here upon the *à priori* grounds of the method. Generally it may be said that the solution of a question in the Theory of Probabilities depends upon the possibility of mentally constructing the problem from hypotheses which appear, whether as a consequence of our knowledge or as a consequence of our ignorance, to be simple and independent. When the data are the probabilities of simple events, and no conditions are added, the problem is in theory sufficiently easy, the sole difficulty consisting in the calculation of complex combinations. But when the data are the probabilities of compound events, or when the events are connected by absolute conditions expressible in logical propositions, or when both these circumstances are present, the difficulty of the required mental construction becomes greater. If we assume the independence of the simple events from which the compound events according to their expression in language are formed, we meet, first, the difficulty that the number of equations thus formed may be greater or less than that which is requisite to obtain a solution; and, secondly, the far more fundamental difficulty that the conditions under which the solution, supposing it to be obtained, is analytically valid may not coincide with the conditions under which the data are possible. It seems indeed likely—at any rate the evidence of particular examples points uniformly to the conclusion—that any attempts to construct the problem upon hypotheses which, while not involved in the actual data are of the same nature as those data (*i. e.* which might conceivably have resulted as facts of observation from the same experience from which the data were derived), *limit* the problem, and lead to solutions which are analytically valid under conditions narrower than those under which the data are possible.

But the processes of mathematical logic enable us, without any addition to the actual data, to effect the required construction of the problem formally—formally because the hypotheses which are regarded as ultimate and independent in that construction refer to an ideal state of things. The nature of the conceptions employed, and their connexion with the conceptions involved in the actual statement of the problem, are discussed in the paper. It is sufficient to say here that, whatever difficulty there may be in these conceptions as conceptions, there is nothing arbitrary in the formal procedure of thought with which they are connected. The probabilities of the ideal events enter as auxiliary quantities into the process of solution, and disappear by elimination from the final result, but they are throughout treated as probabilities, and combined according to the laws of probabilities. I will only say here that the difficulty which has been felt in the conception of the ideal events appears to me to arise from a misdirected attempt to conceive those events by means of the events in the statement of the problem—the true order of thought being that the events in the statement of the problem are, not indeed in their material character, but as subjects of probability and of relations affecting probability, to be conceived by means of the ideal events.

Now the probabilities which constitute the actual data will in general be subject to conditions in order that they may be derived from actual experience. Those conditions admit of mathematical expression.

Generally, if the events in the data are all or any of them compound, and if $p_1, p_2, \ldots p_n$ represent their probabilities, those quantities will be subject to certain conditions, expressible in the form of linear equations or *inequations*, beside the condition that, as representing probabilities, they must be positive proper fractions. All such conditions of either kind are ultimately expressible in the general form

$$b_1p_1+b_2p_2 \ldots +b_np_n+b \overset{=}{<} 0,$$

the coefficients b_1, b_2, ... b_n, b differing in the different conditions so as to indicate that each of the quantities $p_1, p_2, \ldots p_n$ varies between a system of inferior limits expressed by linear functions of the other quantities, and a system of superior limits also so expressed.

Thus, if A, B, C represent any simple events, and if p_1 represent the probability of the concurrence of B and C, p_2 that of the concurrence of C and A, p_3 that of the concurrence of A and B, then p_1, p_2, p_3 must, in order that they may be derived from experience, satisfy the conditions

$$p_1 \overset{=}{>} p_2+p_3-1, \quad p_2 \overset{=}{>} p_3+p_1-1, \quad p_3 \overset{=}{>} p_1+p_2-1,$$

as well as the conditions implied in their being positive proper fractions.

On the other hand, the ideal events being by hypothesis simple and independent, the auxiliary quantities which represent their probabilities will be subject to no other condition *à priori* than that of being positive proper fractions—to no other condition *à priori*, because their actual values are determined in the process of solution.

Now the most general results of the analytical investigation are—

1st. That the auxiliary quantities representing the probabilities of the ideal events admit of determination as positive proper fractions, and, further, of a single definite determination as such, precisely when the original data supply the conditions of a possible experience.

2ndly. That as a consequence of this the probability sought will always lie within such limits as it would have had if determined by actual observation from the same experience as the data.

The proof of these propositions rests upon certain general theorems relating to the solution of a class of simultaneous algebraic equations, and, auxiliary to this, to the properties of a functional determinant.

The following are the principal of those theorems :—

1st. If the elements of any symmetrical determinant are all of them linear homogeneous functions of certain quantities a_1, a_2, . . . a_r—if the coefficients of these quantities in the terms on the principal diagonal of the determinant are all positive—and if, lastly, the coefficients of any of these quantities in any row of elements are proportional to the corresponding coefficients of the same quantity in any other row, then the determinant developed as a rational and integral function of the quantities a_1, a_2, . . . a_r will consist wholly of positive terms.

And, as a deduction from the above,

2ndly. If V be a rational and entire function of any quantities $x_1, x_2, \ldots x_n$, involving, however, no powers of those quantities, and all the coefficients being constant, and if in general V_i represent the sum of those terms in V which contain x_i as a factor, and V_{ij} the sum of those terms in V which contain the product $x_i x_j$, then the determinant

$$\left|\begin{array}{ccccc} V & V_1 & V_2 & \ldots & V_n \\ V_1 & V_{11} & V_{12} & \ldots & V_{1n} \\ V_2 & V_{21} & V_{22} & \ldots & V_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ V_n & V_{n1} & V_{n2} & \ldots & V_{nn} \end{array}\right|$$

will on development consist wholly of positive terms.

3rdly. The definitions being as above, and the function V being in form complete, *i. e.* containing all the terms which by definition it can contain, the system of simultaneous equations

$$\frac{V_1}{V}=p_1, \quad \frac{V_2}{V}=p_2, \ldots \frac{V_n}{V}=p_n,$$

in which $p_1, p_2, \ldots p_n$ represent positive proper fractions, will admit of one, and only one, solution in positive integral values of $x_1, x_2, \ldots x_n$.

4thly. The function V being incomplete in form, *i. e.* wanting some of the terms which it might by definition contain, the system of equations

$$\frac{V_1}{V}=p_1, \quad \frac{V_2}{V}=p_2, \ldots \frac{V_n}{V}=p_n$$

will admit of one, and only one, solution in positive integral values of $x_1, x_2, \ldots x_n$, provided that $p_1, p_2, \ldots p_n$, beside being proper fractions, satisfy certain conditions depending upon the actual form of V.

These conditions are expressible by linear equations or inequations of the general form,

$$b_1 p_1 + b_2 p_2 \ldots + b_n p_n + b \overset{=}{>} 0.$$

In the application to the theory of probabilities, the form of the function V depends upon the explicitly determined logical connexion of the events in the data; the equations or inequations of condition correspond to the conditions of possible experience as a source of the data.

It appears, therefore, that, quite independently of any question of the validity of the logical, and I ought perhaps to add philosophical, grounds of the method, it is a perfect method of interpolation. The analytical investigation, however, shows that, for the mere purpose of interpolation, the process might be modified by altering the coefficients of V without affecting its form ; but it indicates at the same time that such modifications have no definite analogy with that process by which weight is assigned to astronomical observations, and, from their arbitrary character, lead to results which cannot properly be regarded as expressions of probability in any sense.

XVI. "On Simultaneous Differential Equations in which the number of Variables exceeds by more than unity the number of the Equations." By GEORGE BOOLE, Esq., F.R.S. Received June 19, 1862.

(Abstract.)

This paper contains the proof, with some applications, of a method described in a paper bearing nearly the same title which was published in the 'Proceedings of the Royal Society' for March 6, 1862.

XVII. "On the Calculus of Symbols."—Third Memoir. By W. H. L. RUSSELL, Esq., A.B. Communicated by Professor STOKES, Sec. R.S. Received June 18, 1862.

(Abstract.)

The following paper is a continuation of the two preceding Memoirs on the same subject. It has a fourfold object. In the first place, I calculate the general values of the coefficients in the Binomial Theorem given in the first Memoir. In the next place, I give an expression for the form of the coefficient of the general term of the multinomial theorem as previously explained. I then give a theorem for the multiplication of symbolical factors emanating from each other after a given law ; and lastly, I investigate a binomial theorem, reciprocal to the binomial theorem already considered.